CERTAIN STUDIES ON ADVANCE SUMMABILITY METHODS AND THEIR APPLICATIONS IN APPROXIMATION THEORY

RESEARCH PROPOSAL

&

PROPOSED PLAN

for

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CERTAIN STUDIES ON ADVANCE SUMMABILITY METHODS AND THEIR APPLICATIONS IN APPROXIMATION THEORY

1. INTRODUCTION:

A set of numbers $u_1, u_2, ...u_n$ arranged in a specific order is called a sequence if for very positive integer n there exist a number u_n and denoted by $\{u_n\}$.

sIn Mathematics, a series is the sum of the terms of a sequence of numbers. If the number of terms in a series is finite then the series is called finite series and is given by

$$\sum_{k=1}^{n} u_k = u_1 + u_2 + ... + u_n$$

and if number of terms is infinitely large then the series is said to be infinite series and defined by

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + ... + u_n + ...$$

If

$$\sum_{k=1}^{n} u_k = u_1 + u_2 + ... + u_n = s_n$$

then s_n is called n^{th} partial sum of the series $\sum_{k=1}^{n} u_k$.

The series $\sum_{k=1}^{\infty} u_k$ is said to be convergent if the partial sum s_n tends to finite limit s when $n \to \infty$ and if $\lim_{n\to\infty} s_n = \pm\infty$ then $\sum_{k=1}^{\infty} u_k$ is said to be divergent.

If $\sum_{k=1}^{\infty} u_k$ does not converge, but also does not diverge then it is said to be oscillatory.

Some examples of convergent series.

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ...$$

$$1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + ...$$

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} = ...$$

Some examples of divergent series.

1- 2+3 -4+.....

1- 2+4- 8+.....

1 - 1! + 2! -3!+.....

1 + 1+ 1+ +.....

The reciprocals of the positive integers produce a divergent series $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+.....$

1- 1+1-1+..... is a oscillatory series.

ORIGIN OF SUMMABILITY THEORY:

Convergence describes the limiting behaviour to an infinite sequence and series, towords some limit.

Divergence may be some kind of oscillation, unrestricted growth (recognized in the case of an infinite limit) or chaotic behaviour. An infinite series which is divergent can not be used for meaningful computations of its value. Divergent series does not has limit. Therefore, it has almost reached the vanishing point during the last decade of that period. The gradual development of one present day notions of rigor was perhaps the dominant influence. At present time the definitions of convergence and divergence are at common places of elementry analysis. Newton and Leibnitz and all the great mahematicians of the seventeenth and eighteenth centuries, may seem to have manipulated series, knew well enough whether the series which they used were convergent.

Newton and Leibnitz were the first mathematicians to use infinite series systematically, had little temptation to use divergent series. The temptation became greater as analysis widened and it was soon found that they were useful.

The concept of convergence of infinite series was first rigorously formulated by Augustin-Luis Cauchy in 1821 in his course d'Analyse Algebrique and Abel's [1] research on the binomial series in 1826. In the beginning of nineteenth century it was found that there were several series in which the concept of convergence was clearly failed, so it becomes necessary to consider generalize convergence methods. A theory of divergent series was formulated explicitly for the first time in 1890 when Ceàsro [7]

published a paper on the multiplication of series. The extent to which the study of divergent series was continued, both in England and Germany, during the first sixty years of the nineteenth century, has been clearly pointed out by Burkhardt [6] in his paper in 1911.

Since then the theory of series whose sequence of partial sums oscillates has been the centre of creative activity for the leading mathematicians. It was only towards the closing decade of the 19th and in the early years of the 20th century that satisfactory methods were devised, so as to associate with Cauchy's concept of convergence certain values of which may be called their sums in a less direct way. Such processes of summation of series, which were formally tabooed as divergent, have given rise to modern rigorous theory of summability. Summability itself has become an independent branch of mathematics.

EFFECT OF SUMMABILITY TECHNIQUES:

Consider the series

$$\sum(-1)^n = 1\text{-}1\text{+}1\text{-}1\text{+}\ldots\ldots$$

We can find the sum of this series in three ways:

(i) s =(1-1)+(1-1)+(1-1)+..... = 0

(ii) s =1+(-1+1)+(-1+1)+..... = 1+ 0+0+...

= 1

(iii) s =1 - 1 + 1 – 1 +...

s = + 1 – 1 + 1 – 1 +...

2s = 1

thus s =½

One can easily get confused what the sum is. And whether the series is convergent, divergent or oscillatory.

Now the conclusion is that the given series is divergent, since it loses its sum in the usual sense.

We can find its sum in another way.

The sequence of its nth partial sums is

$$s_n \equiv 1, 0, 1, 0.....$$

has so far been rejected as divergent because its partial sum does not approaches a single definite number i.e. osillates unceasingly between 1 and 0. This suggests the idea of forming the arithmetic mean

$$s_n' = \frac{s_0 + s_1 + ... + s_n}{n+1} \quad , n = 0, 1, 2,.....$$

Since $s_n = \frac{1}{2}\left[1 + (-1)^n\right]$ we find that

$$s_n' = \frac{(n+1) + \frac{1}{2}\left[1 + (-1)^n\right]}{2(n+1)}$$

$$= \frac{1}{2} + \frac{1 + (-1)^n}{4(n+1)}$$

so that $$\lim_{n\to\infty} s_n' = \frac{1}{2}$$

A seies for which s_n tends to a limit is said to be summable by arithmetic means or by cesàro's means of first order.

Let us consider another example

$$1 - 2 + 3 - 4 + ...$$

To find the Cesaro sum of this series if it exists, we compute the arithmetic means of partial sums of the series. The partial sums are

$$1, -1, 2, -2, 3, -3,.....$$

and the arithmetic means of these partial sums are

$$1, 0, 2/3, 0, 3/5, 0, 4/7,.....$$

This sequence of means does not converge so, 1-2+3-4+....is **not Cesaro summable.**

Also the sequence of partial sums 1,-1,2,-2,...does not tend towards any finite limit.

Many of these methods easily assign to 1-2+3-4+... a sum ¼. Thus cesaro method is one of the few methods that does not sum 1-2+3-4+...

So, for such series a slightly stronger method, such as Abel summation is required.

Nörlund summability (N, p_n) of Fourier series and conjugate Fourier series was discussed by number of researchers like Pati [29]and Rajagopal [33] etc.

Generalised Nörlund summability (N,p,q) of Fourier series and its conjugate series was discussed by Khare [18] in 1990. (N,p,q) summability reduces to (N,p_n) summability for $q_n = 1\ \forall n$.

K^{λ} - summability was first introduced by Karamata [15]. Thereafter Lotosky [21] considered a particular case $\lambda = 1$. Only after the study of Agnew [3] has done an intensive study of similar methods. Vuĉkoviĉ [42] applied this method for summability of Fourier series. After his result, Kathal [16], Vuĉkoviĉ [42], Ojha [26], Tripathi and Lal [41] have produced their work on K^{λ}-summability.

After a good amount of work done using single summability methods, different product summability methods were put on sound foundation till the beginning of 21st century.

APPROXIMATION THEORY:

Approximation theory is concerned with how functions can best be approximated with simpler functions or the mathematical study of how given quantities can be approximated by other (usually simpler) ones under appropriate conditions. Approximation theory also studies the size and properties of the error introduced by approximation.

APPLICATIONS:

Summability theory can also be applied on other fields of mathematics like functional analysis, approximation theory and applied mathematics, so it is important to point out applications of summability methods in other branches of mathematics and also the development of other fields in mathematics by using tools from summability.

2. PURPOSE OF STUDY:

In this work we shall study the degree of approximation by karamata summability method. The concept of K^{λ} - summability was introduced by Karamata [15], Lotosky [21] reintroduced the special case $\lambda = 1$. Only after the study of Agnew [3], an intensive study of these and similar cases took place. Vuĉkoviĉ [42] applied this method for summability of Fourier series. Kathal [16] extended the result of Vuĉkoviĉ [42]. Working in the same direction Ojha [26], Tripathi and Lal [41] have studied K^{λ}-summability of Fourier

series under different conditions and have generalized the result of Karamata. The purpose of our work is to establish a theorem on degree of approximation by Karamata summability method.

Till the beginning of 21st century, the concept of product summability methods were put on sound foundation and few work has been done so far. The purpose of our work is to study different insinite series by product summability methods.

Fourier series and conjugate Fourier series have been studied by a number of researchers like Singh [37], Khare [18], Mittal and Kumar [24], Singh and Singh [38], Pandey [28] and Jadiya [14] using different single summability methods. The purpose of work is to study Fourier series by $(N, p_n)(E,q)$ product summability method. Our theorem extends several known results on single summability.

The degree of approximation of functions belonging to Lipα, Lip (α, r), Lip (ξ (t) , r) classes has been determined by a number of researchers like Bernstein [4], Alexits [2], Sahney and Geol [35], Qureshi and Neha [32], Chandra [8], Khan [17], Leindler [20] and Rhoades [34] etc. After a quite good number of work on degree of approximation of function by different summability means of its Fourier series for the first time Qureshi [30,31] discussed the degree of approximation of conjugate of a function belonging to Lipschitz class and Lip (α, r) by (N, p_n) means of conjugate Fourier series. We propose to determine the degree of approximation of function belonging to Lipschitz class, $Lip(\xi(t),r)$ and weighted class by $(E,1)(N, p_n)$ product means of Fourier series.

Lal [19] and Nigam [25] discussed $(C,1)(E,1)$ summability of Fourier series. We shall prove a theorem on $(N,p,q)(E,1)$ summability of a Fourier series. $(N,p,q)(E,1)$ summability includes $(N, p_n)(E,1)$ and $(C,1)(E,1)$ summabilities as particular cases.

We shall also study the errors approximation by using different summability methods. The degree of approximation of functions belonging to Lipα, Lip (α, r), Lip (ξ (t), r) and W (L_r, ξ(t)) class will be studied by matrix summability methods. Matrix summability method included Cesàro, Harmonic, Nörlund, generalized Nörlund methods as particular cases.

Rajagopal [33] previously proved a theorem on the Nörlund summability of Fourier series. Result of Rajagopal [33] contains various results due to Hardy [10], Hirokowa

[12], Hirokowa and Kayashima [13], Pati [29], Siddiqi [36], Mishra[23] and Singh [37]. Sharma [39] proved a theorem, dealing with the harmonic summability of double Fourier series. Therefore in this work we shall also prove quite new theorem on double summability of double conjugate Fourier series.

3. PROPOSED WORK:

We propose to investigate the following problems:

1. Degree of approximation of a function belonging to $Lip(\xi(t), r)$ class by (N, p, q) (E, 1) product means of Fourier series.
2. A study on degree of approximation by Karamata summability method.
3. Approximation of functions belonging to different classes of functions by $(E,1)(N, p_n)$ product means.
4. On approximation of functions belonging to different classes by $(N, p_n)(E, q)$ product means.
5. A study on $(N, p_n)(E, q)$ product summability of Fourier series
6. On (E, 1)(C, 1) summability of Fourier series and its conjugate series.
7. Degree of approximation of a function belonging to $Lip(\xi(t), r)$ class by $(C,1)(E, q)$ means of Fourier series.
8. On double summability of double conjugate Fourier series.
9. Degree of approximation of a function belonging to $Lip(\xi(t), r)$ class by (E, 1) (C, 1) Product Means.
10. A study on $(N, p_n)(E, q)$ product summability.
11. Approximation of conjugate of functions belonging to $Lip\alpha$ class and weighted $(L_r, \xi(t))$ class by product means of conjugate series of Fourier series.

4. LITERATURE SURVEY:

Literature is being studied from following books/e-books:

1. Trigonometric Series by Antony Zygmund.
2. Classical and modern methods in summability by Johann Boos, F. Peter Cass.

3. Real Analysis by H. L. Royden.
4. Principles of Mathematical Analysis by Walter Rudin.
5. Fourier Analysis and Approximation of Functions by Trigub, Roald M.
6. Divergent series by G. H. Hardy.
7. Studies on divergent series and summability by Walter Burton Ford.

We have brought relevant papers from Del-net. We are also consulting the research papers published in the journals/database and some of them are listed below.

1. International Journal of Pure and Applied Mathematics
2. Tamkang Journal of Mathematics
3. Ultra Scientist of Physical Sciences
4. Surveys in Mathematics and its Applications
5. Proceeding of American Mathematical Society
6. communications in Applied Analysis
7. Quarterly Journal of Mathematics
8. Science direct
9. Scopus
10. Del-Net
11. MathSciNet
12. Zentralblatt MATH Database
13. CSA Technology Research Database
14. Science Navigator Database
15. Bulletin of The Brazilian Mathematical Society
16. IAENG international Journal of pure and Applied Mathematics.

5. METHODOLOGY:

Some of the most familiar methods of summability and which we shall be concerned in the sequel are those that are known as methods of Abel, Cesàro, K^{λ}, uniform matrix, Nörlund summability, uniform Nörlund summability. It may, however, be mentioned that all these methods can be derived from two basic general process which are termed as –

1. T - processes

2. ϕ - processes

T - Methods are based on formulations of an auxiliary sequence $\{t_m\}$, defined by the sequence-to-sequence transformation, such that

$$t_m = \sum_n C_{m,n} s_n \qquad (m = 0, 1, 2.....)$$

where s_n is the n^{th} partial sum of a given series $\sum u_n$.

The matrix $\|T\| = (C_{m,n})$, in which $C_{m,n}$ is the element in the m^{th} row and n^{th} column, is usually called the matrix of T or a Teoplitz matrix.

The ϕ-methods are based upon the formation of a functional transform $t(x)$ defined by a sequence-to-function transformation

$$t(x) = \sum_n \phi_n(x)\, s_n$$

or, more generally, by the integral transformation

$$t(x) = \int_0^\infty \phi(x, y)\, s(y)\, dy$$

where x is a continuous parameter and the function $\phi_n(x)$ *or* $[\phi(x, y)]$ is defined over a suitable interval of x or [x and y].

A series $\sum u_n$ or the sequence $\{s_n\}$ is said to be convergent if

$$\lim_{n\to\infty} s_n = s$$

By analogy, a series $\sum u_n$ is said to be summable by T-method (or ϕ-method) to sum s if

$$\lim_{m\to\infty} t_m = s \left[\text{ or } \lim_{x\to a} t(x) = s \right]$$

where s is a suitable number.

Let $T = (a_{n,k})$ be an infinite triangular matrix satisfying the Silverman Toeplitz [40] condition of regularity, i.e.

$$\sum_{k=0}^{n} a_{n,k} \to 1 \quad as \quad n \to \infty$$

$$a_{n,k} = 0 \quad for \quad k > n$$

and $\sum_{k=0} |a_{n,k}| \le M$, a finite constant.

Let $\sum_{m=0}^{\infty} u_m$ be an infinite series such that

$$s_k = u_0 + u_1 + \ldots + u_k = \sum_{m=0}^{k} u_m$$

i.e. s_k denotes k^{th} partial sum of series $\sum_{m=0}^{\infty} u_m$.

The sequence-to-sequence transformation

$$t_n = \sum_{k=0}^{n} a_{n,k}\, s_k$$

$$= \sum_{k=0}^{n} a_{n,n-k}\, s_{n-k}$$

defines the sequence $\{t_n\}$ of matrix means of the sequence $\{s_n\}$ generated by the sequence of coefficients $(a_{n,k})$. The series $\sum u_n$ is said to be summable to the sum s_n by matrix method. If $\lim_{n \to a} t_n$ exists and is equal to s (Zygmund [43]) and we write $t_n \to s(T)$, as $n \to \infty$.

Seven important particular cases of matrix means are:

1. (C, 1) means, when $a_{n,k} = \dfrac{1}{(n+1)}$.
2. Harmonic mean when $a_{n,k} = \dfrac{1}{(n-k+1)\log n}$.
3. (C, δ) mean when $a_{n,k} = \dfrac{\binom{n-k+\delta}{\delta-1}}{\binom{n+\delta}{\delta}}$.
4. (H, P) means when $a_{n,k} = \dfrac{1}{\log(p-1)(n+1)} \prod_{q=0}^{p-1} \log q(k+1)$.
5. Nörlund mean when $a_{n,k} = \dfrac{p_{n-k}}{p_n}$ where $p_n = \sum_{k=0}^{n} p_k$.
6. Riesz mean $(\overline{N}, p_n)$ when $a_{n,k} = \dfrac{p_k}{p_n}$.
7. Generalized Nörlund mean (N, p, q) when $a_{n,k} = \dfrac{p_{n-k}\, q_k}{R_n}$ where $R_n = \sum_{k=0}^{n} p_k\, q_{n-k}$.

FOURIER SERIES:

Let $f(x)$ be a 2π - periodic function and integrable over $[-\pi,\pi]$ in the sense of Lebesgue. Then the Fourier series associated with $f(x)$ at a point x is defined by

$$f(x)\sim \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

where the coefficients a_0, a_n and b_n are called Fourier coefficients and given by

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx,\quad n=1,2,3,\ldots.$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx,\quad n=1,2,3,\ldots.$$

CONJUGATE FOURIER SERIES:

Conjugate series of Fourier series (1) is defined by

$$\sum_{n=1}^{\infty}(a_n\sin nx-b_n\cos nx)=\sum_{n=1}^{\infty}B_n(x)$$

DOUBLE FOURIER SERIES:

Let f(x, y) be a function of (x, y), periodic with respect to x and with respect to y, in each case with period 2π and summable in the square Q $(-\pi,-\pi;\ \pi,\ \pi)$. The double Fourier series of a function f(x, y) is given by

$$f(u,v)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\lambda_{m,n}\left[\begin{matrix}\alpha_{m,n}\cos mx\cos ny+\beta_{m,n}\sin mx\cos ny+\gamma_{m,n}\cos mx\sin ny\\+\delta_{m,n}\sin mx\sin ny\end{matrix}\right]$$

$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\lambda_{m,n}A_{m,n}(x,y)\qquad(1)$$

and

$$\alpha_{m,n}=\frac{1}{\pi^2}\iint_Q f(x,y)\cos mx\cos ny\,dx\,dy$$

with three similar expressions for $\beta_{m,n}$, $\gamma_{m,n}$ and $\delta_{m,n}$, where Q represents the fundamental square $(-\pi,-\pi;\ \pi,\ \pi)$.

One can associate three conjugate series to the double Fourier series (1) in the following way:

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\lambda_{m,n}\left[-\beta_{m,n}\cos mx\cos ny+\alpha_{m,n}\sin mx\cos ny-\delta_{m,n}\cos mx\sin ny+\gamma_{m,n}\sin mx\sin ny\right] \quad (2)$$

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\lambda_{m,n}\left[-\gamma_{m,n}\cos mx\cos ny-\delta_{m,n}\sin mx\cos ny+\alpha_{m,n}\cos mx\sin ny+\beta_{m,n}\sin mx\sin ny\right] \quad (3)$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\lambda_{m,n}\left[\delta_{m,n}\cos mx\cos ny-\gamma_{m,n}\sin mx\cos ny-\beta_{m,n}\cos mx\sin ny+\alpha_{m,n}\sin mx\sin ny\right] \quad (4)$$

where $\lambda_{m,n}=1$, $\lambda_{m,0}=\lambda_{0,n}=\dfrac{1}{2}$, $m,n\geq 1$, $\lambda_{0,0}=\dfrac{1}{4}$.

DOUBLE CONJUGATE FOURIER SERIES:

The conjugate functions $\tilde{f}^{(1)}(x,y)$, $\tilde{f}^{(2)}(x,y)$ and $\tilde{f}(x,y)$ corresponding to (2), (3) and (4) are defined as

$$\tilde{f}^{(1)}(x,y)=-\frac{1}{\pi}\int_0^{\pi}\frac{f(x+s,y)-f(x-s,y)}{2\tan\frac{1}{2}s}\,ds$$

$$\tilde{f}^{(2)}(x,y)=-\frac{1}{\pi}\int_0^{\pi}\frac{f(x,y+t)-f(x,y-t)}{2\tan\frac{1}{2}t}\,dt$$

$$\tilde{f}(x,y)=-\frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\left[\frac{f(x+s,y+t)-f(x-s,y+t)-f(x+s,y-t)+f(x-s,y-t)}{2\tan\frac{1}{2}s\ \ 2\tan\frac{1}{2}t}\right]ds\,dt$$

TYPES OF SUMMABILITY METHODS:

We are going to use single and product summability methods and some of them are described below.

(E, 1) SUMMABILITY:

Let $\sum_{n=0}^{\infty}u_n$ be a given infinite series with sequence of its partial sums s_n.

If $$E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}s_k \to s \quad \text{as } n \to \infty$$

Then the series $\sum_{n=0}^{\infty} u_n$ is said to summable to S by (E, 1) summability and E_n^1 is called the (E, 1) transform of s_n .

NÖRLUND SUMMABILITY:

Let $p_m \geq 0$ be a sequence of positive terms where $P_n = p_0 + p_1 + ... + p_n$.

Define t_m such that $t_m = \dfrac{p_m s_0 + p_{m-1}s_1 + ... + p_0 s_m}{p_0 + p_1 + ... + p_n}$

If $t_m \to \infty$ when $m \to \infty$ and $s_n = u_0 + u_1 + ... + u_n$,
then we write

$$s_n \to s, \quad \sum u_n = s(N, p_n).$$

GENERALIZED NÖRLUND SUMMABILITY:

Let $\{p_n\}$ and $\{q_n\}$ be the sequences of constants, real or complex, such that

$$P_n = p_0 + p_1 + p_2 + ... + p_n = \sum_{k=0}^{n} p_k \to \infty, \; as \; n \to \infty$$

$$Q_n = q_0 + q_1 + q_2 + ... + q_n = \sum_{k=0}^{n} q_k \to \infty, \; as \; n \to \infty$$

$$R_n = p_0 q_n + p_1 q_{n-1} + p_2 q_{n-2} + ... + p_n q_0 = \sum_{k=0}^{n} p_k q_{n-k} \to \infty, \; as \; n \to \infty$$

Now convolution of two sequences $\{p_n\}$ and $\{q_n\}$ is defined as

$$R_n = (p * q)_n = \sum_{k=0}^{n} p_{n-k} q_k \, .$$

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series with the sequence of its n^{th} partial sums $\{s_n\}$.

We write $$t_n^{p,q} = \frac{1}{R_n}\sum_{k=0}^{n} p_{n-k} q_k s_k$$

If $R_n \neq 0$, for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$.

If $t_n^{p,q} \to s, as\ n \to \infty$, then the series $\sum_{n=0}^{\infty} u_n$ or sequence $\{s_n\}$ is summable to s by generalized Nörlund method (Borwein [5]) and is denoted by

$$s_n \to s(N, p, q)$$

Particular cases of (N, p, q) mean are:

1. (N, p, q) mean reduces to (N, p_n) summability mean if $q_n = 1, \forall n$.
2. (N, p, q) mean reduces to $(\overline{N}, q_n)$ if $p_n = 1, \forall n$.

(C, 1) SUMMABILITY:

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series. Then $\sum_{n=0}^{\infty} u_n$ is said to be summable (C, 1) to the sum s if

$$\lim_{n\to\infty} \frac{s_0 + s_1 + \ldots\ldots + s_n}{n+1} \to s.$$

$(N, p_n)(C,1)$ SUMMABILITY:

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series. Then $\sum_{n=0}^{\infty} u_n$ is said to be summable (C, 1) to the sum s if

$$\lim_{n\to\infty} \frac{s_0 + s_1 + \ldots\ldots + s_n}{n+1} \to s.$$

Let $\{p_n\}$ be the sequence of real constants such that

$$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty, as\ n \to \infty$$

then (N, p_n) transform (Hardy [10]) of the $(C,1)$ transform defines the $(N, p_n)(C,1)$ transform of the partial sum s_n of series $\sum_{n=0}^{\infty} u_n$. If

$$(NC)_n^1 = \frac{1}{n+1} \sum_{k=0}^{\infty} \frac{1}{p_k} \sum_{\nu=0}^{k} p_{k-\nu} s_\nu \to s, as\ n \to \infty$$

where C_n^1 denotes the (C, 1) transform of s_n then $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(N, p_n)(C,1)$ mean.

$(N, p_n)(E,1)$ SUMMABILITY:

Let

$$E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}s_k \to s \quad \text{as } n \to \infty$$

Then the series $\sum_{n=0}^{\infty}u_n$ is said to summable to S by (E, 1) summability and E_n^1 is called the (E, 1) transform of s_n .

Let $\{p_n\}$ be the sequence of real constants such that

$$P_n = \sum_{\nu=0}^{n} p_\nu \to \infty, \; as \; n \to \infty$$

The (N, p_n) transform (Hardy [10]) of the $(E,1)$ transform defines the $(N, p_n)(E,1)$ transform of the partial sum s_n of series $\sum_{n=0}^{\infty}u_n$. If

$$(NE)_n^1 = \frac{1}{P_n}\sum_{k=0}^{n} p_k \frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}s_\nu \to s, \; as \; n \to \infty$$

where E_n^1 denotes (E, 1) transform of s_n then $\sum_{n=0}^{\infty}u_n$ is said to be summable by $(N, p_n)(E,1)$ mean.

$(C,1)(E,1)$ SUMMABILITY:

The $(C,1)$ transform of the $(E,1)$ transform E_n^1 defines the $(C,1)(E,1)$ transform of the partial sum s_n of series $\sum_{n=0}^{\infty}u_n$.

Let $\sum_{n=0}^{\infty}u_n$ be a given infinite series. Then $\sum_{n=0}^{\infty}u_n$ is said to be summable (C, 1) to the sum s if

$$\lim_{n\to\infty}\frac{s_0 + s_1 + \ldots\ldots + s_n}{n+1} \to s .$$

Thus if

$$(CE)_n^1 = \frac{1}{n}\sum_{k=0}^{n}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}s_\nu \to s, \; as \; n \to \infty .$$

where E_n^1 denotes (E, 1) transform of s_n i.e.

$$E_n^1 = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}s_k \to s \quad \text{as } n \to \infty$$

and s is a finite constant then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(C,1)(E,1)$ mean.

KARAMATA SUMMABILITY:

Let us define, for n = 0, 1, 2, ….., the numbers $\begin{bmatrix} n \\ m \end{bmatrix}$, *for* $0 \le m \le n$, *by*

$$\prod_{\nu=0}^{n-1}(x+\nu) = \sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix} x^m = \frac{\Gamma(x+n)}{\Gamma(x)}$$

$$= x(x+1)(x+2).....(x+n-1)$$

The numbers $\begin{bmatrix} n \\ m \end{bmatrix}$ are known as the absolute value of stirling number of first kind.

Let $\{s_n\}$ be the sequence of partial sums of an infinite series $\sum u_n$ and let us write

$$s_n^{\lambda} = \frac{\Gamma(\lambda)}{\Gamma(\lambda+n)}\sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}\lambda^m s_m$$

to denote the n^{th} K^{λ}- mean of order $\lambda > 0$. If $s_n^{\lambda} \to s \; as \; n \to \infty$, where s is a fixed finite number then the sequence $\{s_n\}$ or the series $\sum u_n$ is said to be summable by Karmata method (K^{λ}) of order $\lambda > 0$ to the sum s and we can write

$$s_n^{\lambda} \to s\,(K^{\lambda}) \; as \; n \to \infty.$$

We define the norm by

$$\|f\|_p = \left\{\int_0^{2\pi} |f(x)|^p \, dx\right\}^{\frac{1}{p}} \quad p \ge 1$$

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial t_n of order n under sup norm $\| \|_{\infty}$ is defined by (Zygmund [43])

$$\|t_n - f\|_\infty = \sup\{|t_n(x) - f(x)| : x \in R\}$$

and $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = \min_{t_n} \|t_n - f\|_r$$

This method of approximation is called trigonometric Fourier approximation (TFA).

A function $f \in Lip\alpha$ if

$$|f(x+t) - f(x)| = O\left(|t^\alpha|\right) \text{ for } 0 < \alpha \le 1$$

and

$f(x) \in Lip(\alpha, r)$ for $0 \le x \le 2\pi$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O\left(|t^\alpha|\right), 0 < \alpha \le 1, r \ge 1$$

(Definition 5.38 of Mc Fadden [22])

Given a positive increasing function $\xi(t)$ and $r \ge 1$, $f(x) \in Lip(\xi(t), r)$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O(\xi(t))$$

and $f \in W\left(L^r, \xi(t)\right)$

$$\left(\int_0^{2\pi} \left|\{f(x+t) - f(x)\}\sin^\beta x\right|^r dx\right)^{\frac{1}{r}} = O(\xi(t)), \beta \ge 0$$

$\beta = 0$, our newly defined class $W(L_r, \xi(t))$ reduces to the class $Lip(\xi(t), r)$.

6. CONCLUSION:

Complete work will come up in form of Ph. D. Thesis.

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